Two Equations to the Stars -
Part Two: The Photon Sail Equation

How to explain photon sail maths to mid-secondary school students

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There are currently two feasible ways to propel a space probe to the stars - fusion rockets and laser sails - each governed by a simple equation.

In the last issue, P38, we explained the first of these equations, governing rockets, and how middle secondary school students can be introduced to the mathematics, physics and engineering behind all rockets, including the ones that can propel us to the stars. In this second part we explain the simple equation for laser and solar sail propulsion.

Interstellar propulsion theorist Robert Forward. Credit (image and caption): *A Photon Beam Propulsion Timeline*, *www.centauri-dreams.org/2016/06/24/a-photon-beam-propulsion-timeline/*

$$\alpha = \frac{2\eta P}{Mc}$$

The Photon Sail Equation

Who is this article for?

Like the first, this article is primarily intended for teachers introducing school students to space physics, maths and engineering in mid-secondary school (middle school in USA) and their students. The objective is to give these students an accessible yet fairly rigorous understanding of the fundamentals of photon propulsion. It also introduces students to the consequences of the equation, and physics and maths behind it. The equation governs the propulsion challenges of low mass probes such as i4is Project Dragonfly (2014) and the ongoing Project Starshot financed by Breakthrough Initiatives (breakthroughinitiatives.org/initiative/3). These ideas look like our best route to early interstellar probes.

▶

1 Reading order

Teachers - read parts *2-Introduction*, *3-The Equation* and as much of the rest as you can easily understand. If the approach is interesting then get in touch with i4is via info@i4is.org to discuss how we can take this further for you and your students. If it is not interesting then tell i4is why so we can refine it (address as above).

School students - read parts *2-Introduction*, *5-The Problem*, *6-The Story* and *8-For School Students - What you can achieve*. If you feel confident then read - from *3-The Equation* as far as you understand and then report progress to your maths, science or engineering teacher and ask them to take a look at this article. If you don't feel confident then ask a teacher to help with any difficult maths, science or engineering.

For both teachers and school students, get in touch at any time with questions, suggestions and requests for help - always info@i4is.org.

2 Introduction

This is the second of two articles for Principium explaining how to understand the equations which define our ability to reach our outer Solar System and the stars. Our aim in this article is to provide a way to understand Robert Forward's photon sail equation without using mathematics beyond basic secondary school level. It is intended for teachers, self-motivated secondary school students and interested people in general. More about its objectives in section 8 - *For School Students - What you can achieve*.

The photon sail equation was introduced by visionary physicist and engineer, Robert Forward, in his paper *Roundtrip interstellar travel using laser-pushed lightsails* [1].

The i4is Project Dragonfly logo (Credit: Adrian Mann). The project name Dragonfly refers to Dr Forward's novel, {i}*The Flight of the Dragonfly*, Timescape Books, 1984 (en.wikipedia.org/wiki/Rocheworld) about an interstellar mission using a laser driven light sail propulsion system.

3 The Equation

Dr Forward's equation works for anything pushed by photons including laser light, microwaves and the light of the Sun.

Dr Forward advocated -

> "...beamed-power propulsion in that the "engines" of the vehicle are left behind in the solar system and the power and reaction mass are transmitted out to the rest of the vehicle that carries the payload. This system will use large solar-pumped lasers to convert sunlight into coherent radiation, large lightweight optics to transmit the coherent laser beam over interstellar distances, and large lightweight reflective sails carrying the payload that are pushed by the momentum of the reflected laser photons."

This is close to the idea applied in two early i4is projects -

• Project Dragonfly:

Project Dragonfly: A feasibility study of interstellar travel using laser-powered light sail propulsion, Perakis et al [2].

Project Dragonfly: Sail to the stars, Häfner et al [3].
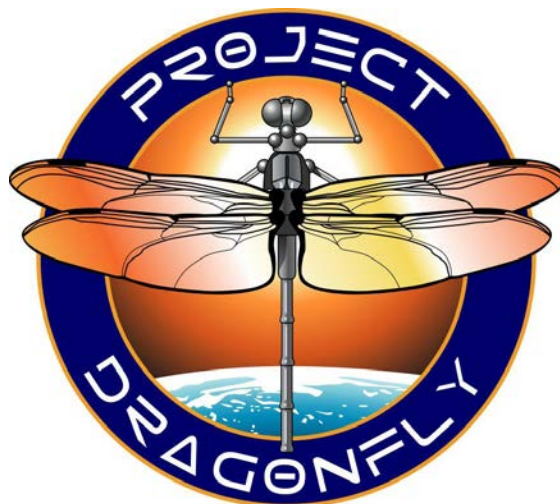
[1] *Roundtrip Interstellar Travel Using Laser-Pushed Lightsails,* Robert L Forward Hughes Research Laboratories, 1984, Journal of Spacecraft and Rockets, Vol. 21, No. 2, p. 187 - 195. web.archive.org/web/20170808080011id_/http://interstellar-flight.ru/design/base_e/rit-1.pdf

[2] *Project Dragonfly: A feasibility study of interstellar travel using laser-powered light sail propulsion*, Nikolaos Perakis, Lukas Schrenk, Johannes Gutsmiedl, Artur Koop, Martin J Losekamm - all Technical University of Munich (TUM), www.nperakis.com/_files/ugd/b85f56_76b3eb456a3f4ebeb5e819ca6c682367.pdf

[3] *Project Dragonfly: Sail to the stars* Tobias Häfner (Université Paul Sabatier Toulouse - UPST, Manisha Kushwaha (Cranfield University, UK), Onur Celikb (Cranfield), Filippo Bellizzi (Cranfield), Acta Astronautica, V154, January 2019, Pages 311-319. www.researchgate.net/profile/Onur-Celik-2/publication/317491721_Dragonfly_Sail_to_the_Stars/links/593bb891a6fdcc17a9c89270/Dragonfly-Sail-to-the-Stars.pdf

• Project Andromeda:

The Andromeda Study: A Femto-Spacecraft Mission to Alpha Centauri, Hein et al, 2017[1].
It shares the basis for the major Breakthrough Starshot programme - with the variation that Starshot currently aims to use lasers based on Earth rather than in space.

Project Andromeda logo

Credit: Adrian Mann
i4is.org/what-we-do/technical/andromeda-probe/

Visualisation 100 gigawatt scale laser beamer. Credit: Breakthrough Initiatives, *breakthroughinitiatives.org/concept/3, i4is.org/what-we-do/technical/andromeda-probe/*

Dr Forward gives his equation as follows -
"The acceleration of a vehicle of mass M and reflectance η driven by an incident laser power P is -
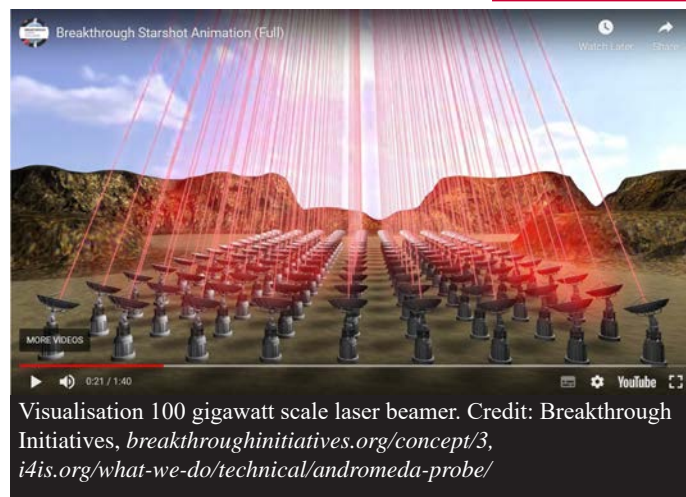
$$\alpha = \frac{2\eta P}{Mc}$$

- where c is the velocity of light and the factor 2 comes from the double momentum transfer to the sail by the reflected photons."

What this means is that more reflectance and more laser power gives more acceleration but more mass gives less acceleration. The units of this are -
α acceleration in metres per second per second.
η (Greek letter eta) is dimensionless - it's just the ratio of reflected to absorbed light - so, with the massive laser powers we need for missions like this it needs to be very close to 1 which is perfect reflection. P is the power of the photons in watts (joules of energy per second). M is the mass of the vehicle to be accelerated in kilograms.
The problem here is, c, the velocity of light, an unchangeable constant of 100,000 kilometres per second or 100,000,000 metres per second in MKS units. This very large value means projects like this need to use extremely powerful photon sources and to keep the mass of the vehicle very small.

4 The Implications

We have a specific objective, an interstellar probe. Let's compare Dr Forward's photon sail equation with the rocket equation of his predecessor, Tsiolkovsky (for more about him and his rocket equation see *Two Equations to the Stars - Part One: The Rocket Equation* - in the last issue of Principium, P37 [2]. This also shows how Al-Khwarizmi and Isaac Newton taught us things about algebra and physics which are relevant here too).
So here are the two equations side by side -

Tsiolkovsky	Forward
$\Delta V = Ve \ln(Mo / Mf)$	$\alpha = 2\eta P / Mc$

On the left ΔV is the change in velocity produced by using all the rocket propellant, Ve is the exhaust velocity of the rocket, Mo is the original mass of the rocket including the propellant, Mf is the final mass of the rocket when all the propellant has been expelled and ln is the natural logarithm function (log to the base **e** = 2.71828 - rather than 10).
On the right is Dr Forward's equation as before.

> **What is ΔV (DeltaV)?**
>
> ΔV is "change in velocity". It's important in both our equations because of what we are seeking from a rocket or a laser sail - speed. We only care about accelaration because it delivers ΔV but to get that speed we don't care if it's a big acceleration for a short time or a small accelaration for a long time.

[1] *The Andromeda Study: A Femto-Spacecraft Mission to Alpha Centauri*, Andreas M Hein et al, August 2017. arxiv.org/abs/1708.03556
[2] Principium 37 i4is.org/principium-37/, *Two Equations to the Stars - Part One: The Rocket Equation* i4is.org/wp-content/uploads/2022/05/Two-Equations-to-the-Stars-part-one-Principium37-AW-2205261003-opt.pdf

Now let's make the equation comparable, with the same units on the left side of both equations-writing Dr Forward's equation with ΔV to the left of the "=". The two equations are now easily comparable -

Tsiolkovsky	Forward
$\Delta V = Ve \ln(Mo / Mf)$	$\Delta V = 2\eta Pt / Mc$

ΔV is the change in velocity on both left and right. On the left it's produced by using all the rocket propellant. As before, Ve is the exhaust velocity of the rocket and the ratio of original and final masses Mo/Mf, the mass ratio. As before ln is the natural logarithm function.

On the right ΔV is the change in velocity produced by the photons hitting the sail, η remains the reflectivity of the sail, P the power of the photons hitting the sail, c is the velocity of light and the factor 2 comes from the double momentum transfer to the sail by the reflected photons. **The added component is t - the time for which the photons hit the sail.**

This already has interesting consequences -

• **Tsiolkovsky** tells us not to worry about how long his rocket operates for. In space this is literally true - using up your fuel quickly gives you no advantage so a rocket which pushes gently for a long time is as good, in terms of velocity added, as one which pushes harder for a shorter time. It's only the exhaust velocity (often converted into specific impulse, I_{sp} [1]) and the ratio of the initial and final masses which make the difference.

• **Forward** tells us not to worry about propellant (since we are not carrying any) and only think about photon power and reflectivity and worry about vehicle mass. Again no worry about how long his sail is pushed for; a little push for a long time is as good as a big push for a short time. The snag here is that the photon source, a laser or the Sun, is fixed and the vehicle is moving away from it so the inverse square law applies, double the distance and divide the power by four.

This leads to hard engineering judgements. You will see this in any discussion of either of these two known ways of achieving the enormous velocities we need to get to the stars in any reasonable time.

[1] I_{sp} is measured in seconds, a ratio of the thrust produced to the mass flow of the propellant(s). This comes out as (metres/sec) / (metres/sec²) , velocity/acceleration so the units are just seconds. A good high I_{sp} propellant example is Xenon, typical I_{sp} 1500-4000 secs, 8 times as much I_{sp} as LOx+LH₂, but not enough thrust to liftoff from Earth.

[2] *Untersuchungen über die Druckkräfte des Lichtes* (*Investigations on the pressure forces of light*) - see - https://it.wikisource.org/wiki/Scientia_-_Vol._VII/Die_Druckkr%C3%A4fte_des_Lichtes (German text)

5 The Problem

The problem with rockets (or reaction propulsion generally) for interstellar flight is the same using photon sailing (or externally driven propulsion generally). It's a long way to anywhere really interesting (with apologies to all the scientists studying the interstellar medium, ISM). How do we get there in any reasonable time -

▪ less than a lifetime?
▪ less than the duration of European exploration of the Earth?
▪ or (even) less than the duration of recorded human history?

Recall that our fastest and furthest probes, the two Voyagers, won't reach the distances required for tens of thousands of years.

Pyotr Nikolaevich Lebedev (1866 –1912). Credit: Wikipedia/slovari.yandex.ru

Thinking about this led to the idea of sailing as long ago as 1899 by Russian physicist P N Lebedev quoting James Clerk Maxwell -

"...the rays of light must exert pressure on those bodies on which they fall. The fact that this property of light escaped the researchers was easily explained by the exceptionally low behaviour of these forces: Maxwell calculated (1873) that in the clear sky at lunchtime [he was British!] the compressive force of solar radiation on four square metres hardly equals the weight of one-tenth of a gram."

- Lebedev's ideas were taken up by the ubiquitous pathfinder for our move into space, Konstantin Tsiolkovsky. He wrote in his paper *The Spaceship*, 1924-1926-

"...the third and most attractive method of obtaining speed. This is to transmit energy to the vehicle from outside, from the earth. The vehicle itself need not store up material energy (that is, ponderable energy - in the form of explosive or fuel). The energy is transmitted to it from the planet in the form of a parallel beam of electromagnetic rays of short wave-length.

◄

...This parallel beam of electric or even light (solar) rays should exert pressure by itself (there can be no doubt that such pressure exists [citing Lebedev]) such pressure can give the vehicle a sufficient speed. In that case, one would not need any supplies for ejection. **The last method would seem to be the most refined."** [1] (my **emphasis**)

More about this early thinking in *Tsiolkovsky - Interstellar Pioneer*, Principium 20, February 2018 (i4is.org/wp-content/uploads/2018/02/Principium20.pdf).
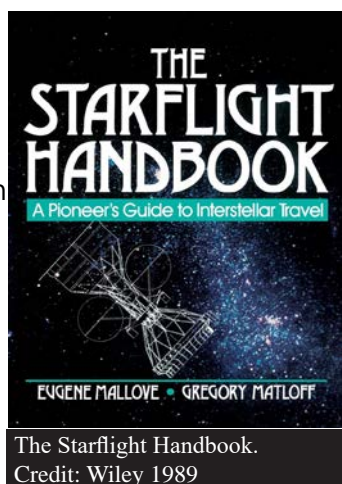
6 The Story

Our story begins again with Newton for physics and Al-Khwarizmi for algebra - and I will assume knowledge of the basics of these as explained in the first of these two Principium articles. So, where does Forward's photon sail equation come from? Forward does not give a derivation or cite a source.

A good starting point is in a book by Mallove and Matloff (M&M), *The Starflight Handbook* [2]. Back in 1989 M&M told us (see their Technical Note 5-2) that the energy in a photon is given by **E=hv** and the momentum of a photon is implied by λ**=h/p** [3].

The Starflight Handbook. Credit: Wiley 1989

In these two equations E is the energy in joules, v is the light frequency in Hz, p is the momentum of the photon in kilogram (kg) metres per second, λ (Greek lambda) is the wavelength in metres and h is Planck's constant in metres squared kilograms per second. Let's re-write that with f as the frequency since v is easily confused with velocity - and transposing the second equation to put momentum of the left side. This gives us E=hf and p=h/λ. But M&M point out that the velocity of a wave is the frequency times the wavelength so the velocity of light c=fλ.

So what is the momentum of our photon?

With p=h/λ and λ=c/f (transposing that third equation) so p=hf/c and because E=hf then p=E/c. Now if the photon is bounced off something, like a sail, then it transfers twice its momentum so a perfectly reflecting solar sail has momentum added p=2E/c. But we want to find the acceleration and since momentum is just mass times velocity so mv=2E/c and so v=2E/mc. Acceleration is **rate of change of velocity** so, with energy in photons per second, we get ΔV = 2Et/cm where t is the time during which the photons are hitting the sail. If our sail is not a perfect reflector we need to include reflectivity η giving ΔV = 2Etη/cm which is Forward's equation in a version comparable with the Tsiolkovsky rocket equation (ie including the time for which the photons hit the sail).

7 The Consequences

Let's compare these two propulsion techniques - mainly as they affect interstellar missions-

	Reaction (rocket)	Photon (sail)
Limitation	Exhaust velocity, mass ratio	1/c means tiny probes, massive laser power requirement
Capability	Fusion required but not yet attained	Very high probe velocity
Difficulties	Massive vehicle demands in-space construction	Tiny probes limit downlink bandwidth and science payload

Both techniques require technologies which could become weapons - fusion rocket exhaust and multi-gigawatt lasers respectively. The issues extend beyond the simple list above and there is a lot of published work examining them. See 10 *Further resources*, below.

If you spot a major missing issue or you struggle to understand one of the issues then get in touch with i4is, info@i4is.org, and raise it with us.

[1] In - *K E TSIOLKOVSKY Selected Works*, compiled by V N Sokolsky, General editor Acad. A A Blagonravov, translated from the Russian by G Yankovsky, University Press of the Pacific, Honolulu, Hawaii, 1968-2004 - the publisher has vanished and URL universitypressofthepacific.com is defunct but the book may still be available.
[2] *The Starflight Handbook: A Pioneer's Guide to Interstellar Travel*, Eugene F Mallove, Gregory L Matloff, Wiley, 1989.
[3] How can a photon, with no mass, have momentum? Here Einstein helps with his famous $E=mc^2$ - so mass is just energy and since photons have energy they have something that can act as mass. The amounts are very small because c, the velocity of light, is very large so a little energy, as in a photon, has that something that can act as mass but in tiny quantities.

►

IKAROS. Credit: JAXA Institute of Space and Astronautical Science
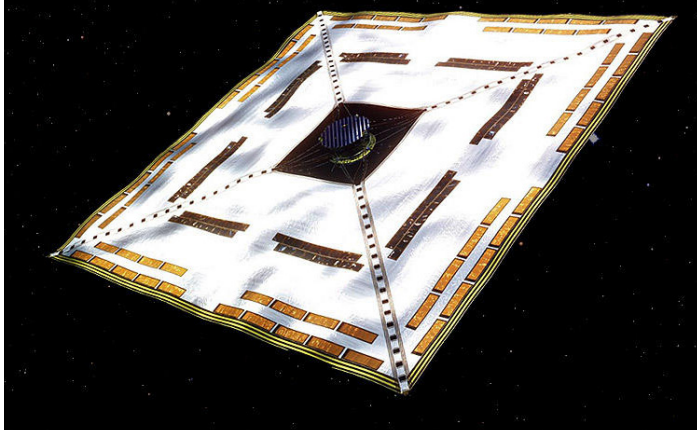
8 For School Students - What you can achieve

The basic physics behind Forward's equation is even simpler than Tsiolkovsky's equation. Tsiolkovsky had to deal with the fact that a rocket loses mass as it uses up its propellant. He used calculus, specifically integration, to account for this. As the first of these articles explained, you can "work around" this using the perfectly mathematically respectable technique of numerical approximation. So a bit of simple algebra and a spreadsheet using only addition and multiplication gives an equivalent result. The light sail equation derivation above, 6 *The Story*, does use the concept of momentum. If you don't yet understand how that works any maths or science teacher can help - find the one you comunicate with best!

With Forward's simple equation you can work out how fast any sailcraft will fly. This includes implemented solar sail missions like Japan's *IKAROS [1]* and the Planetary Society's *LightSail* [2], small laser-propelled demonstrators like the i4is Glowworm proposal [3], early ideas like the i4is

Lightsail 2. Credit: The Planetary Society, *www.planetary.org/sci-tech/lightsail*

Dragonfly work [4] and Andromeda study [5] - and all the way to the mighty 100 gigawatt thinking of Breakthrough Starshot [6].

From there you can move on to the economics of ground based laser banks, as in Starshot, and space based lasers, as in Andromeda. This leads to the issues of public safety of big laser banks, the political or private support you will need for your interstellar thinking and your own attitude and commitment (or not?) to the cause of sending our first probe to the stars. It all starts from the physics and the engineering in this article but that is just the beginning.

[1] *IKAROS Small Scale Solar Powered Sail Demonstration Satellite*, www.isas.jaxa.jp/en/missions/spacecraft/current/ikaros.html
[2] *LightSail, a Planetary Society solar sail spacecraft,* www.planetary.org/sci-tech/lightsail
[3] *Project Glowworm*, i4is.org/what-we-do/technical/project-glowworm/
[4] *Project Dragonfly,* i4is.org/what-we-do/technical/project-dragonfly/ and en.wikipedia.org/wiki/Project_Dragonfly_(space_study) - for a deeper dive into this work see the papers cited above 3 *The Equation* under Project Dragonfly
[5] *Andromeda Probe*, i4is.org/what-we-do/technical/andromeda-probe/
[6] *Breakthrough Starshot*, breakthroughinitiatives.org/initiative/3 and en.wikipedia.org/wiki/Breakthrough_Starshot

9 For Teachers - The Vision

This article and its predecessor aim to inspire both school students and their teachers. We at i4is believe that space can inspire both and that interstellar is the ultimate expression of that outward urge which has driven our species from early times. Looking at interstellar concepts challenges science, maths and engineering in many ways. Propulsion is just one of them but is perhaps the most simple and obvious in its relevance, the easiest to approach mathematically. Yet it contains some of the most extreme physics and challenging engineering - and ultimately maths too.
i4is has numerous resources and capabilities to assist you in handling issues arising from interstellar thinking which range from "hard" science and engineering to economic, moral and sometimes even political problems. Take a look at our website i4is.org. Don't hesitate to contact us via address info@i4is.org - always monitored.

10 Further resources

There is a simple "ready reckoner" on the i4is website, *Exploring Equations* (i4is.org/equations/) which shows the results of inserting values into the terms of both the rocket equation and the photon sail equation. You can meet our Technical team (i4is.org/what-we-do/technical/) and our Education team (i4is.org/what-we-do/education/). Lots of links to ongoing photon sail work in *8 For School Students - What you can achieve*, above. If you are interested in getting more deeply into the maths, science and engineering then search scholar.google.com using terms such as -

Search terms	Example result
interstellar sail	Robert Forward's classic *Roundtrip interstellar travel using laser-pushed lightsails*, 1984, has gathered 281 citations - take a look at some of them but not all!
starshot sail	Kevin Parkin's *The breakthrough starshot system model* (arxiv.org/abs/1805.01306) is a particularly useful starting point. It dates from 2018 and has 105 citations so take a look there too.
tsiolkovsky photon	NASA paper *The Physics of Solar Sails*, 2002 (ntrs.nasa.gov/api/citations/20030093608/downloads/20030093608.pdf)

11 Conclusion

I hope to have shown both teachers and school students that the underlying maths of propulsion for interstellar travel is accessible without profound mathematical expertise - and that it leads on to wider considerations both within and beyond space science and technology. Please get in touch if you have comments, questions or suggestions to add or change any of this.
At info@i4is.org we are always listening.

The Vision: Going Soon?   Here is David Hardy's 1972 idea - with his own explanation -
Proxima 1972: The most effective extrasolar scenes are often those with a red sun, whether a red supergiant or a dwarf. One of the latter is our closest stellar neighbour, Proxima Centauri, a small member of the Alpha Centauri triple system. This was painted, in acrylics, for The Challenge of the Stars with Patrick Moore in 1972. In order to have liquid water, the planet would need to orbit the red star in 10 days. The constellation Cassiopeia can be seen, with an extra star - our Sun. Credit: David Hardy, www.astroart.org